

Navier–Stokes limit of elastic bilinear kinetic equations

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Introduction

Collisional kinetic equation with no force :

$$(\partial_t + v \cdot \nabla_x)F(t, x, v) = \mathcal{C}[F(t, x, \cdot)](v)$$

Macroscopic elastic conservation (mass, momentum, energy):

$$\varphi(v) = 1, v_1, \dots, v_d, |v|^2, \quad \int_{\mathbb{R}^d} \mathcal{C}[F](v) \varphi(v) dv = 0.$$

Microscopic dissipation (H-theorem) : For some convex $\Phi : \mathbb{R} \rightarrow \mathbb{R}$

$$\forall F, \quad \int_{\mathbb{R}^d} \mathcal{C}[F](v) \Phi(F(v)) dv \leq 0.$$

and, denoting $\mathbf{m} = \int_{\mathbb{R}^d} F(v) (1, v, |v|^2) dv$ the conserved macroscopic quantities

$$\forall F, \quad \int_{\mathbb{R}^d} \mathcal{C}[F](v) \Phi(F(v)) dv = 0 \Leftrightarrow \mathcal{C}[F] = 0 \Leftrightarrow F(v) = \mathcal{M}_{\mathbf{m}}(v).$$

Examples : (quantum) Boltzmann and Landau, BGK, non-linear Fokker-Planck

Introduction

Small fluctuations in diffusive regime : $F(t, x, v) = \mathcal{M}_{1,0,1}(v) + \varepsilon f^\varepsilon (\varepsilon^2 t, \varepsilon x, v)$

$$\partial_t f^\varepsilon = \varepsilon^{-2}(\mathcal{L} - \varepsilon v \cdot \nabla_x) f^\varepsilon + \varepsilon^{-1} \mathcal{Q}(f^\varepsilon, f^\varepsilon) + o(\varepsilon^{-1})$$

where $\mathcal{C}[\mathcal{M}_{1,0,1} + \varepsilon g] = \varepsilon \mathcal{L}g + \varepsilon^2 \mathcal{Q}(g, g) + o(\varepsilon^2)$

Theorem (Bardos, Golse, Levermore, 1991 (for $\Phi(u) = -\log u$))

If $f^\varepsilon \rightarrow f^0$ and every term of the equation converges formally, then

$$f^0(t, x, v) = (u^0(t, x) \cdot v + \theta^0(t, x)(|v|^2 - cst.)) \mathcal{M}_{1,0,1}(v)$$

$$\begin{cases} \partial_t u^0 + u^0 \cdot \nabla u^0 = \kappa_{inc} \Delta u^0 + \nabla p, & \nabla \cdot u^0 = 0, & \kappa_{inc} = \kappa_{inc}(\mathcal{L}) > 0, \\ \partial_t \theta^0 + u^0 \cdot \nabla \theta^0 = \kappa_{Fou} \Delta \theta^0, & \kappa_{Fou} = \kappa_{Fou}(\mathcal{L}) > 0 \end{cases}$$

Weak convergence: For Boltzmann (Bardos, Golse, Levermore, Saint-Raymond, Arsénio)

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Quantitative convergence: For Boltzmann or Landau (Bardos, Ukai, Gallagher, Tristani, Carrapatoso, Rachid) using spectral studies (Ellis, Pinsky, Yang, Yu)

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Limitations on quantitative approach:

- ▶ Only $\Phi(u) = -\log u$ (exclude quantum equations)
- ▶ Equations treated separately (although common structure)
- ▶ Non-constructive spectral studies
- ▶ Restrictive integrability condition

Introduction

Our structural assumptions

Consider $V^1 \subset V \subset V^{-1}$ where $V := L^2(\mu(v)^{-1}dv)$ and $V^{-1} := (V^1)'$

(H1) Isotropy : $\mathcal{L}$ and $\mathcal{Q}$ commute with orthogonal change of variables.

(H2) Macroscopic conservation law:

$$\ker \mathcal{L} = \text{Span}(\mu, v_1\mu, \dots, v_d\mu, |v|^2\mu) \subset V^1$$

$$\langle \mathcal{L}f, \varphi \rangle_V = \langle \mathcal{Q}(f, f), \varphi \rangle_V = 0, \quad \varphi \in \ker(\mathcal{L})$$

(H3) Microscopic dissipation (linearized H-theorem):

$$\langle \mathcal{L}f, f \rangle_V \leq -\|\Pi_{(\ker \mathcal{L})^\perp} f\|_{V^1}^2 \quad \text{and} \quad [\mathcal{L}, \langle v \rangle^{1 \text{ or } 2}] = \text{L.O.T.}$$

$$\mathcal{L} = \text{dissipative} + \text{compact}$$

(H4) Control of the collisions by the energy V and the dissipated entropy V^1 :

$$\|\mathcal{Q}(f, g)\|_{V^{-1}} \lesssim \|f\|_{V^1} \|g\|_V + \|f\|_V \|g\|_{V^1}$$

(H3) and (H4): gain and loss of “regularity” $V^{\pm 1}$

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$$\langle \mathcal{Q}(f, g), h \rangle_V \lesssim \|h\|_{V^1} (\|f\|_{V^1} \|g\|_V + \|f\|_V \|g\|_{V^1})$$

(H3) and (H4): gain and loss of “regularity” $V^{\pm 1}$

Strategy

Our problem: profile of small fluctuations f^ε around an equilibrium

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathcal{L} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon, f^\varepsilon) + \cancel{o(\varepsilon^{-1})}, \quad f^\varepsilon(0, x, v) = f_{\text{in}}(x, v)$$

Goal: show that $\exists! f^\varepsilon$

$$f^\varepsilon(t, x, v) \xrightarrow{\varepsilon \rightarrow 0} \left(u(t, x) \cdot v + \theta(t, x) (|v|^2 - \text{cst.}) \right) \mu(v)$$

$$\begin{cases} \partial_t u + u \cdot \nabla_x u = \kappa_{\text{inc}} \Delta_x u, & \nabla_x \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla_x \theta = \kappa_{\text{Fou}} \Delta_x \theta. \end{cases}$$

and describe/quantify the convergence.

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$$f^\varepsilon(t) = U^\varepsilon(t)f_{\text{in}} + \Psi^\varepsilon(f^\varepsilon, f^\varepsilon)(t)$$

where

$$U^\varepsilon(t) := \exp\left(\frac{t}{\varepsilon^2}(\mathcal{L} - \varepsilon v \cdot \nabla_x)\right)$$

$$\Psi^\varepsilon(f, f)(t) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - \tau) \mathcal{Q}(f(\tau), f(\tau)) d\tau$$

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Strategy for the existence of solutions

Find a decomposition of the semigroup/nonlinearity:

$$U^\varepsilon(t) = \exp\left(\frac{t}{\varepsilon^2} (\mathcal{L} - \varepsilon v \cdot \nabla_x)\right) = U_{\text{hydro}}^\varepsilon \oplus U_{\text{kin}}^\varepsilon$$

$$\Psi^\varepsilon(f, f)(t) = \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - \tau) \mathcal{Q}(f(\tau), f(\tau)) d\tau = \Psi_{\text{hydro}}^\varepsilon(f, f) + \Psi_{\text{kin}}^\varepsilon(f, f)$$

compatible with some “hydrodynamic” and “kinetic” regimes/norms:

$$\begin{aligned} \|U_{\text{hydro}}^\varepsilon f\|_{\mathcal{H}} &\lesssim \|f\|_{H_x^\ell V_v}, & \|U_{\text{kin}}^\varepsilon f\|_{\mathcal{K}^\varepsilon} &\lesssim \|f\|_{H_x^\ell V_v} \\ \|\Psi_{\text{hydro}}^\varepsilon(f, f)\|_{\mathcal{H}} &\lesssim \|f\|_{\mathcal{H}}^2, & \|\Psi_{\text{kin}}^\varepsilon(f, f)\|_{\mathcal{K}^\varepsilon} &\lesssim \|f\|_{\mathcal{K}^\varepsilon}^2 \end{aligned}$$

where the regimes are defined as

$$\|f\|_{\mathcal{H}}^2 := \sup_{t \geq 0} \|f(t)\|_{H_x^\ell \mathbf{V}_v^1}^2 + \int_0^\infty \|\nabla_x f(t)\|_{H_x^\ell \mathbf{V}_v^1}^2 dt, \quad \begin{array}{l} \text{(heat equation)} \\ \text{(Navier-Stokes)} \end{array}$$

$$\|f\|_{\mathcal{K}^\varepsilon}^2 := \sup_{t \geq 0} e^{2\sigma t/\varepsilon^2} \|f(t)\|_{H_x^\ell V_v}^2 + \frac{1}{\varepsilon^2} \int_0^\infty e^{2\sigma t/\varepsilon^2} \|f(t)\|_{H_x^\ell \mathbf{V}_v^1}^2 dt, \quad \begin{array}{l} \text{(dissipative)} \\ \text{equation} \end{array}$$

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$$\|U_{\text{hydro}}^\varepsilon f\|_{\mathcal{H}} \lesssim \|f\|_{H_x^\ell V_v}, \quad \|U_{\text{kin}}^\varepsilon f\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{H_x^\ell V_v}$$

$$\|\Psi_{\text{hydro}}^\varepsilon(f, f)\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}^2, \quad \|\Psi_{\text{kin}}^\varepsilon(f, f)\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{\mathcal{K}^\varepsilon}^2$$

and seek a solution of the form $f^\varepsilon = f_{\text{hydro}}^\varepsilon + f_{\text{kin}}^\varepsilon + \dots$ solving the (arbitrary) system

$$\begin{cases} f_{\text{hydro}}^\varepsilon(t) = U_{\text{hydro}}^\varepsilon(t) f_{\text{in}} + \Psi_{\text{hydro}}^\varepsilon(f_{\text{hydro}}^\varepsilon, f_{\text{hydro}}^\varepsilon) + \text{small coupling} \\ f_{\text{kin}}^\varepsilon(t) = U_{\text{kin}}^\varepsilon(t) f_{\text{in}} + \Psi_{\text{kin}}^\varepsilon(f_{\text{kin}}^\varepsilon, f_{\text{kin}}^\varepsilon) + \text{small coupling} \\ \dots \end{cases}$$

Strategy

Strategy for the convergence

Kinetic convergence : $\|f_{\text{kine}}^\varepsilon\|_{\mathcal{K}^\varepsilon} \lesssim 1 \xrightarrow{\text{def}} f_{\text{kine}}^\varepsilon = \mathcal{O}\left(e^{-\sigma t/\varepsilon^2}\right)$

Hydrodynamic convergence : To prove the convergence of

$$f_{\text{hydro}}^\varepsilon(t) = U_{\text{hydro}}^\varepsilon(t) f_{\text{in}} + \Psi_{\text{hydro}}^\varepsilon(f_{\text{hydro}}^\varepsilon, f_{\text{hydro}}^\varepsilon) + \dots$$

(1) Show the convergence of the “hydrodynamic” operators:

$$U_{\text{hydro}}^\varepsilon = U_{\text{NS}}^0 + \mathcal{O}(\varepsilon) + \text{dispersive} \quad \text{and} \quad \Psi_{\text{hydro}}^\varepsilon(f, f) = \Psi_{\text{NS}}^0(f, f) + \mathcal{O}(\varepsilon)$$

(2) Check from **explicit** expressions of U_{NS}^0 and Ψ_{NS}^0 that

$$\begin{aligned} f(t) &= U_{\text{NS}}^0(t) f(0) + \Psi_{\text{NS}}^0(f, f)(t) \\ \Leftrightarrow \begin{cases} f(t, x, v) = (u(t, x) \cdot v + \theta(t, x) (|v|^2 - \text{cst.})) \mu(v), \\ (u, \theta) \text{ is a mild solution of INSF} \end{cases} \end{aligned}$$

Outline of the proof of the (non)linear bounds

Proposition (G, Lods)

The semigroup and Duhamelized nonlinearity split

$$U^\varepsilon(t) = \exp\left(\frac{t}{\varepsilon^2}(\mathcal{L} - \varepsilon v \cdot \nabla_x)\right) = U_{\text{hydro}}^\varepsilon(t) \oplus U_{\text{kine}}^\varepsilon(t)$$

$$\frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - \tau) \mathcal{Q}(f(\tau), f(\tau)) d\tau = \Psi_{\text{hydro}}^\varepsilon(f, f)(t) + \Psi_{\text{kine}}^\varepsilon(f, f)(t)$$

and satisfy the continuity estimates

$$\|U_{\text{hydro}}^\varepsilon f\|_{\mathcal{H}} \lesssim \|f\|_{H_x^\ell V_v}, \quad \|U_{\text{kine}}^\varepsilon f\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{H_x^\ell V_v}$$

$$\|\Psi_{\text{hydro}}^\varepsilon(f, f)\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}^2, \quad \|\Psi_{\text{kine}}^\varepsilon(f, f)\|_{\mathcal{K}^\varepsilon} \lesssim \varepsilon \|f\|_{\mathcal{K}^\varepsilon}^2$$

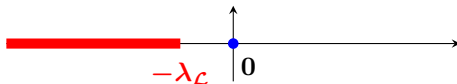
Goals/difficulties:

1. Find the decomposition in U^ε -invariant subspaces $H_x^\ell V_v = \text{hydro}^\varepsilon \oplus \text{kine}^\varepsilon$
2. Compensate the stiff factor $\frac{1}{\varepsilon}$
3. Compensate the deregularizing effect $\mathcal{Q} : V^1 \rightarrow V^{-1}$

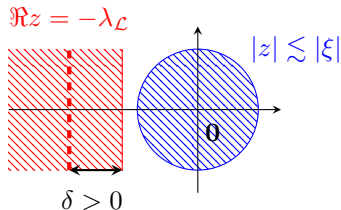
Outline of the proof of the (non)linear bounds

Finding stable subspaces of $\mathcal{L} - v \cdot \nabla_x$: localization of $\Sigma(\mathcal{L} - iv \cdot \xi)$

Localization for $\xi = 0$:

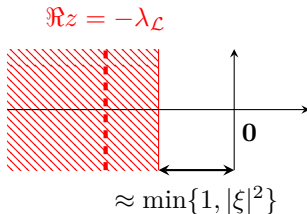


Localization for $|\xi| \ll 1$:



Factorization method inspired from [T]
using $[\mathcal{L}, \langle v \rangle] = \text{L.O.T. from } (\mathbf{H3})$

Localization for $|\xi| \gtrsim 1$:



Mode by mode hypocoercivity from [D]

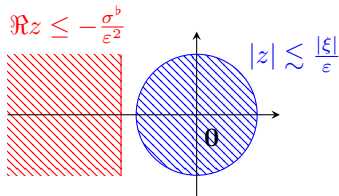
Disjoint parts of $\Sigma(\mathcal{L} - iv \cdot \xi) \Rightarrow$ invariant **kinetic**(ξ) and **hydro**(ξ) subspaces

Outline of the proof of the (non)linear bounds

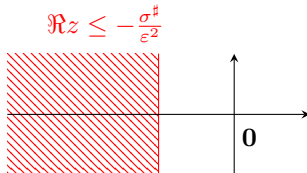
The kinetic regime

$$\text{Scaling } (t, x) \rightarrow \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) \Rightarrow \mathcal{L} - i(v \cdot \xi) \rightarrow \frac{1}{\varepsilon^2} (\mathcal{L} - i\varepsilon(v \cdot \xi))$$

Localization for $|\xi| \ll \varepsilon^{-1}$:



Localization for $|\xi| \gtrsim \varepsilon^{-1}$:



$\xrightarrow[\text{theorem}]{\text{Gearhart-Pruss}}$ exponential decay **uniform in ε** : $f \in V \Rightarrow e^{t \frac{\sigma}{\varepsilon^2}} U_{\text{kin}}^\varepsilon f \in L_t^\infty V_v$

$\xrightarrow[\text{V}^1\text{-dissipation of } \mathcal{L}]{\text{energy method using}}$ integral **regularization uniform in ε** : $\varepsilon^{-1} e^{t \frac{\sigma}{\varepsilon^2}} U_{\text{kin}}^\varepsilon f \in L_t^2 \mathbf{V}_v^1$

$$\Rightarrow \boxed{\|U_{\text{kin}}^\varepsilon f\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{H_x^\ell V_v}} \quad \|f\|_{\mathcal{K}^\varepsilon} = \|e^{\sigma t / \varepsilon^2} f\|_{L_t^\infty H_x^\ell V_v} + \varepsilon^{-1} \|e^{\sigma t / \varepsilon^2} f\|_{L_t^2 H_x^\ell V_v^1}$$

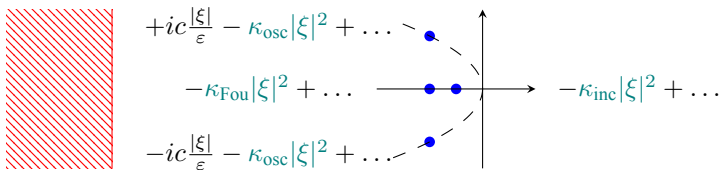
$$\Rightarrow \|\varepsilon^{-1} U_{\text{kin}}^\varepsilon *_t \mathbf{Q}(f, f)\|_{\mathcal{K}^\varepsilon} = \boxed{\|\Psi_{\text{kin}}^\varepsilon(f, f)\|_{\mathcal{K}^\varepsilon} \lesssim \varepsilon \|f\|_{\mathcal{K}^\varepsilon}^2}$$

Study of the hydrodynamic operators

The hydrodynamic regime

Perturb^{tion} theory for $\mathcal{L} - iv \cdot \xi$: conserv^{tion} laws ($\xi = 0$) $\rightarrow$ hydro. modes ($|\xi| \ll 1$)

$$\Re z \leq -\sigma^b / \varepsilon^2$$



$$U_{\text{hydro}}^\varepsilon = U_{\text{inc}}^\varepsilon + U_{\text{Fou}}^\varepsilon + U_{\text{osc}}^\varepsilon, \quad \widehat{U_\star^\varepsilon(t)f(\xi)} = e^{\varepsilon^{-2}t\lambda_\star(\varepsilon\xi)} \mathcal{P}_\star(\varepsilon\xi) \widehat{f}(\xi)$$

- $\Re(\varepsilon^{-2}\lambda_\star(\varepsilon\xi)) \approx -\kappa_\star|\xi|^2 \Rightarrow U_\star^\varepsilon(t) \approx e^{t\kappa_\star\Delta_x} \mathcal{P}_\star(\varepsilon D_x) \Rightarrow$ parabolic in (t, x)
- $\mathcal{P}_\star(\xi) : V^{-1} \rightarrow V^1 \Rightarrow$ regularization in v

$$\boxed{\|U_\star^\varepsilon f\|_{\mathcal{H}} \lesssim \|f\|_{H_x^\ell V_v^{-1}}} \quad \|f\|_{\mathcal{H}} = \|f\|_{L_t^\infty H_x^\ell V_v^1} + \|\nabla_x f\|_{L_t^2 H_x^\ell V_v^1}$$

- $\mathcal{Q} \perp \ker(\mathcal{L}) + \text{expansion of } \mathcal{P}_\star \Rightarrow \mathcal{P}_\star(\varepsilon\xi)\mathcal{Q} = \mathcal{P}_\star(0)\mathcal{Q} + \mathcal{O}(\varepsilon\xi)$ (gain ε)
 $\Rightarrow \|\varepsilon^{-1}U_{\text{hydro}}^\varepsilon *_t \mathcal{Q}(f, f)\|_{\mathcal{H}} = \boxed{\|\Psi_{\text{hydro}}^\varepsilon(f, f)\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}^2}$

Conclusion

Statement of the theorem

Previous estimates: $\|f_{\text{hydro}}^\varepsilon(0)\| \ll 1$ and $\varepsilon \ll 1 \xrightarrow{\text{Picard near } 0} \exists f^\varepsilon$.

Better: $\exists f_{\text{NS}}^0$ + norm such that $\|f_{\text{NS}}^0\|_{\mathcal{H}_{(f_{\text{NS}}^0, \delta)}} \leq \delta \ll 1 \xrightarrow{\text{Picard near } f_{\text{NS}}^0} \exists f^\varepsilon$ (as in [GT])

Theorem (G, Lods) — Annales Henri Lebesgue (10.5802/ahl.215)

Consider any **(non-small)** $f_{\text{in}} \in H_x^\ell L_v^2(\mu^{-1}dv)$, for any $\varepsilon \ll 1$, $\exists!$ solution to

$$\partial_t f^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x f^\varepsilon = \varepsilon^{-2} \mathcal{L} + \varepsilon^{-1} \mathcal{Q}(f^\varepsilon, f^\varepsilon), \quad f^\varepsilon \in L_t^\infty H_x^\ell L_v^2(\mu^{-1}dv dx),$$

with the **same lifespan** as the Navier-Stokes limit f_{NS}^0 , and for smooth initial data

$$f^\varepsilon = f_{\text{NS}}^0 + \mathcal{O}\left(e^{-t\sigma/\varepsilon^2}\right) + \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}\left((\varepsilon/t)^{\frac{d-1}{2}}\right)$$

with **computable constants**, and for incompressible initial data

$$f^\varepsilon = f_{\text{NS}}^0 + \mathcal{O}\left(e^{-t\sigma/\varepsilon^2}\right) + \mathcal{O}(\varepsilon) \quad (\text{optimal})$$

Conclusion

Remarks and perspectives

Better integrability in v : similar assumptions in $W = L^2(m(v)dv) \supset V$:

- ▶ $f_{\text{in}}(x, \cdot) \in W_v \Rightarrow f_{\text{kin}}^\varepsilon(t, x, \cdot) \in W_v$ but $f_{\text{hydro}}^\varepsilon(t, x, \cdot) \in V_v \subset W_v$
- ▶ Nonlinear theory more complicated (Boltzmann without cutoff/Landau):

$$\|\mathcal{Q}(f, g)\|_{W^{-1}} \not\leq \|f\|_W \|g\|_{W^1} + \|f\|_{W^1} \|g\|_W$$

Use of isotropy: adaptable to relativistic velocities $v \cdot \nabla_x \rightarrow \frac{v}{\langle v \rangle} \cdot \nabla_x$

Perspectives:

- ▶ OK for bilinear approximation of general models $\rightarrow$ include remainder $o(\varepsilon^{-1})$:
 - ▶ Quantum Boltzmann/Landau: $\mathcal{C}[F] = \mathcal{Q}(F, F) + \hbar^3 \mathcal{T}(F, F, F)$
 - ▶ BGK: $\mathcal{C}[F] = \mathcal{M}[F] - F$
 - ▶ Nonlinear Fokker-Planck: $\mathcal{C}[F] = T_F \nabla_v \cdot \left(\mathcal{M}[F] \nabla_v \left(\frac{F}{\mathcal{M}[F]} \right) \right)$
- ▶ Confining force $\nabla_x \phi \Rightarrow$ modified transport $v \cdot \nabla_x \rightarrow v \cdot \nabla_x - \nabla_x \phi \cdot \nabla_v$
- ▶ Fluctuation around inhomogeneous Maxwellian (stationary Euler sol. (R, U, T))

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 - ▶ Quantum Boltzmann/Landau: $\mathcal{C}[F] = \mathcal{Q}(F, F) + \hbar^3 \mathcal{T}(F, F, F)$
 - ▶ BGK: $\mathcal{C}[F] = \mathcal{M}[F] - F$
 - ▶ Nonlinear Fokker-Planck: $\mathcal{C}[F] = T_F \nabla_v \cdot \left(\mathcal{M}[F] \nabla_v \left(\frac{F}{\mathcal{M}[F]} \right) \right)$
- ▶ Confining force $\nabla_x \phi \Rightarrow$ modified transport $v \cdot \nabla_x \rightarrow v \cdot \nabla_x - \nabla_x \phi \cdot \nabla_v$
$$\partial_t f^\varepsilon + \varepsilon^{-1} (v \cdot \nabla_x f^\varepsilon - \nabla_x \phi \cdot \nabla_v f^\varepsilon) = \varepsilon^{-2} \underbrace{e^{-\phi} \mathcal{L}}_{\text{degraded coercivity}} + \varepsilon^{-1} e^{-\phi} \mathcal{Q}(f^\varepsilon, f^\varepsilon),$$
- ▶ Fluctuation around inhomogeneous Maxwellian (stationary Euler sol. (R, U, T))

Conclusion

Remarks and perspectives

Better integrability in v : similar assumptions in $W = L^2(m(v)dv) \supset V$:

- ▶ $f_{\text{in}}(x, \cdot) \in W_v \Rightarrow f_{\text{kin}}^\varepsilon(t, x, \cdot) \in W_v$ but $f_{\text{hydro}}^\varepsilon(t, x, \cdot) \in V_v \subset W_v$
- ▶ Nonlinear theory more complicated (Boltzmann without cutoff/Landau):

Use of isotropy: adaptable to relativistic velocities $v \cdot \nabla_x \rightarrow \frac{v}{\langle v \rangle} \cdot \nabla_x$

Perspectives:

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$$F(t, x, v) = \mathcal{M}_{R(\varepsilon x), U(\varepsilon x), T(\varepsilon x)} + \varepsilon f^\varepsilon(\varepsilon^2 t, \varepsilon x, \varepsilon v),$$

Thank you for your attention

Extra

Finding stable subspaces of $\mathcal{L} - v \cdot \nabla_x$: localization of $\Sigma(\mathcal{L} - iv \cdot \xi)$ for $|\xi| \ll 1$

Goal: find $\Re z \geq -\lambda_{\mathcal{L}} + \delta$ such that $(\mathcal{L} - iv \cdot \xi - z)^{-1}$ is bounded on V

Naive approach for localization:

$$(\mathcal{L} - iv \cdot \xi - z)^{-1} = \underbrace{(\mathcal{L} - z)^{-1}}_{\mathcal{O}(1/|z|)} + (\mathcal{L} - iv \cdot \xi - z)^{-1} \underbrace{(iv \cdot \xi)}_{\text{unbounded!}} \underbrace{(\mathcal{L} - z)^{-1}}_{\mathcal{O}(1/|z|)} \quad (\star)$$

Enlargement method: $\mathcal{L} = \mathcal{B} + \mathcal{A}$ with $\mathcal{A} = M \mathbf{1}_{|v| \leq M} : V \rightarrow \langle \cdot \rangle^{-1} V$

$$\begin{aligned} [\mathcal{L}, \langle v \rangle] &= \text{L.O.T. from (H3)} \Rightarrow \|(\mathcal{B} - iv \cdot \xi - z)^{-1}\| \lesssim 1 \text{ on } V \text{ and } \langle \cdot \rangle^{-1} V \\ &\Rightarrow \|(\mathcal{L} - z)^{-1}\| \lesssim 1 + 1/|z| \text{ on } V \text{ and } \langle \cdot \rangle^{-1} V \end{aligned}$$

Insert $(\star)$ in the right hand side of the expansion

$$(\mathcal{L} - iv \cdot \xi - z)^{-1} = (\mathcal{B} - iv \cdot \xi - z)^{-1} + (\mathcal{L} - iv \cdot \xi - z)^{-1} \mathcal{A} (\mathcal{B}_{\xi} - z)^{-1}$$

to use that $\mathcal{A}$ compensates the unboundedness of v :

$$\begin{aligned} (\mathcal{L} - iv \cdot \xi - z)^{-1} &= \underbrace{(\mathcal{B} - iv \cdot \xi - z)^{-1}}_{\text{bounded}} + (\mathcal{L} - z)^{-1} \mathcal{A} (\mathcal{B}_{\xi} - z)^{-1} \\ &\quad + (\mathcal{L} - iv \cdot \xi - z)^{-1} \underbrace{(iv \cdot \xi)(\mathcal{L} - z)^{-1} \mathcal{A} (\mathcal{B}_{\xi} - z)^{-1}}_{\text{of order } |\xi|(1+1/|z|)} \end{aligned}$$

Extra

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$$\|(\mathcal{L} - iv \cdot \xi - z)^{-1}\| (1 - \text{cst.} |\xi| (1 + 1/|z|)) \lesssim 1 + \frac{1}{|z|}$$

Extra

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to use that $\mathcal{A}$ compensates the unboundedness of v :

$$\boxed{|\xi| \ll 1 \text{ and } |\xi| \ll |z| \implies \|(\mathcal{L} - iv \cdot \xi - z)^{-1}\| \lesssim 1}$$

Extra

Finding stable subspaces of $\mathcal{L} - v \cdot \nabla_x$: localization of $\Sigma(\mathcal{L} - iv \cdot \xi)$ for $|\xi| \ll 1$

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$$|\xi| \ll 1 \implies \Sigma(\mathcal{L} - iv \cdot \xi) \subset B(0, C|\xi|) \cup \{\Re z \geq -\lambda_{\mathcal{L}} + \delta\}$$

Extra

Construction of the hydrodynamic modes

Expansion of the spectral hydrodynamic projector/Kato's isomorphism :

Resolvent bounds + enlargement method using $[\mathcal{L}, \langle v \rangle^2 \times] = \text{L.O.T. from (H3)}$

$\Rightarrow$ expansion of hydrodynamic projector $\mathcal{P}(\xi) : V^{-1} \rightarrow \text{hydro space}(\xi) \subset V^1$

$\Rightarrow$ expansion of Kato's isomorphism $\Phi(\xi) : \ker(\mathcal{L}) \rightarrow \text{hydro space}(\xi)$

Diagonalization/expansion of the hydrodynamic part of $\mathcal{L} - iv \cdot \xi$ for $|\xi| \ll 1$:

Define the reduced matrix $L(\xi) \in \mathcal{B}(\ker(\mathcal{L})) \stackrel{\Phi(\xi)}{\sim} (\mathcal{L} - i(v \cdot \xi))|_{\text{hydro. space}(\xi)}$

Isotropy $\Rightarrow L(\xi)|_{|\xi|=0}$ diagonalizable $\Rightarrow$ diagonalization/expansion of $L(\xi)$

$\Rightarrow$ expansion of spectral projections using expansion of $\Phi(\xi)$:

$$\mathcal{P}(\xi) = \mathcal{P}_{\text{Fou}}(\xi) \oplus \mathcal{P}_{\text{inc}}(\xi) \oplus \mathcal{P}_{+\text{osc}}(\xi) \oplus \mathcal{P}_{-\text{osc}}(\xi),$$

$$\mathcal{P}_\star(\xi) = \mathcal{P}_\star^{(0)}(\xi/|\xi|) + \xi \cdot \mathcal{P}_\star^{(1)}(\xi/|\xi|) + \mathcal{O}(|\xi|^2) : V_v^{-1} \rightarrow V_v^1$$

$\Rightarrow$ expansion of eigenvalues:

$$\lambda_{\text{Fou}}(\xi) = -\kappa_{\text{Fou}}|\xi|^2 + \mathcal{O}(|\xi|^3), \quad \lambda_{\text{inc}}(\xi) = -\kappa_{\text{inc}}|\xi|^2 + \mathcal{O}(|\xi|^3),$$

$$\lambda_{\pm\text{osc}}(\xi) = \pm ic|\xi| - \kappa_{\text{osc}}|\xi|^2 + \mathcal{O}(|\xi|^3)$$

Extra

Hydrodynamic regime ($\star = \text{inc, Fou, } \pm \text{osc}$)

Linear estimate:

$$\lambda_{\star}(\xi) \approx i\alpha_{\star}|\xi| - \kappa_{\star}|\xi|^2 \Rightarrow \Re \left(\frac{\lambda_{\star}(\varepsilon\xi)}{\varepsilon^2} \right) \leq -\frac{\kappa_{\star}}{2}|\xi|^2 \quad \text{and} \quad \mathcal{P}_{\star}(\xi) : V_v^{-1} \rightarrow V_v^1$$

$$\Rightarrow \underbrace{\left\| e^{t \frac{\lambda_{\star}(\varepsilon \nabla_x)}{\varepsilon^2}} \mathcal{P}_{\star}(\varepsilon \nabla_x) f \right\|_{H_x^{\ell} V_v^1}}_{\|U_{\star}^{\varepsilon}(t)f\|_{H_x^{\ell} V_v^{-1}}} \lesssim \left\| e^{t \frac{\kappa_{\star}}{2} \Delta_x} f \right\|_{H_x^{\ell} V_v^{-1}} \Rightarrow \boxed{\|U_{\text{hydro}}^{\varepsilon} f\|_{\mathcal{H}} \lesssim \|f\|_{H_x^{\ell} V_v^{-1}}}$$

Reminder: $\mathcal{H} = L_t^{\infty} H_x^{\ell} \cap L_t^2 \dot{H}_x^{\ell+1}$ (parabolic regularity)

Nonlinear estimate:

$$\mathcal{P}^{(0)} \Pi_{\ker(\mathcal{L})} = \mathcal{P}^{(0)} \quad \text{and} \quad \mathcal{Q} \perp \ker(\mathcal{L}) \text{ from } (\mathbf{H2}) \Rightarrow \mathcal{P}^{(0)} \mathcal{Q} = 0$$

$$\Rightarrow \frac{1}{\varepsilon} \mathcal{P}_{\star}(\varepsilon \xi) \mathcal{Q} \approx \mathbf{0} + \xi \cdot \mathcal{P}^{(1)}(\xi/|\xi|) \mathcal{Q}(f, g) \in V_v^1$$

$$\Rightarrow \left\| \frac{1}{\varepsilon} U_{\text{hydro}}^{\varepsilon} *_t \mathcal{Q}(f, f) \right\|_{\mathcal{H}} = \boxed{\left\| \Psi_{\text{hydro}}^{\varepsilon}(f, f) \right\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}^2}$$

Extra

Rough idea of the proof of hydrodynamic convergence

Refining the previous step, one gets for $\star = \text{Fou, inc}$

$$U_{\star}^{\varepsilon}(t) = U_{\star}^0(t) + \mathcal{O}(\varepsilon) \quad \text{and} \quad \Psi_{\star}^{\varepsilon}(t) = \Psi_{\star}^0(t) + \mathcal{O}(\varepsilon)$$

where each limit writes

$$U^0(t) = e^{t\kappa_{\star}\Delta_x} \mathcal{P}^{(0)}(\nabla_x)$$
$$\Psi_{\star}^0(f, f)(t) = \int_0^t e^{(t-\tau)\kappa_{\star}\Delta_x} \mathcal{P}^{(1)}(\nabla_x) \mathcal{Q}(f(\tau), f(\tau)) \, d\tau$$

which is related to the mild formulation of Navier-Stokes-Fourier:

$$u(t) = e^{t\kappa_{\text{inc}}\Delta_x} u(0) + \int_0^t e^{(t-\tau)\kappa_{\text{inc}}\Delta_x} \mathbb{P} [\nabla_x \cdot (u(\tau) \otimes u(\tau))] \, d\tau$$
$$\theta(t) = e^{t\kappa_{\text{Fou}}\Delta_x} \theta(0) + \int_0^t e^{(t-\tau)\kappa_{\text{Fou}}\Delta_x} [\nabla_x \cdot (\theta(\tau)u(\tau))] \, d\tau$$

And for the oscillating terms:

$$U_{\pm\text{osc}}^{\varepsilon}(t) = \mathcal{O}\left(\frac{\varepsilon}{t}\right) \quad \text{and} \quad \Psi_{\pm\text{osc}}^{\varepsilon}(f, f) = \mathcal{O}(\varepsilon).$$

Extra

Kato's reduction process: eigen. pbm. in Banach $\rightarrow$ eigen. pbm. in finite dimension

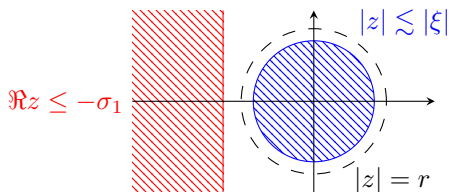


Figure: Localization of $\text{Spec}(\mathcal{L} - i(v \cdot \xi))$ for $|\xi| \ll 1$

Goal: expansion of eigenvalues and eigenfunctions of $(\mathcal{L} - i(v \cdot \xi))|_{\text{hydro. space}(\xi)}$

Difficulty: $\text{hydro. space}(\xi)$ depends on ξ

Extra

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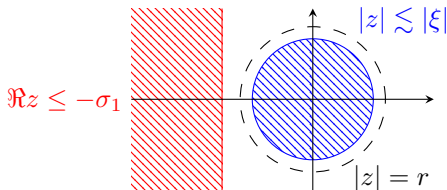


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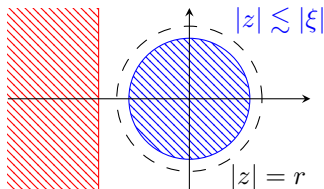
Difficulty: $\text{hydro. space}(\xi)$ depends on ξ

Solution: Rectify $\mathcal{L} - i(v \cdot \xi)$ to a matrix $L(\xi)$ by conjugating with

$$\Phi(\xi) : \text{hydro. space}(\xi) \xrightarrow{\text{iso.}} \ker \mathcal{L} \approx \mathbb{C}^{d+2}$$

Extra

Kato's reduction process



Projection on the hydrodynamic spectrum(ξ) = $R(\mathcal{P}(\xi))$):

$$\mathcal{P}(\xi) = \frac{1}{2i\pi} \oint_{|z|=r} (z - \mathcal{L} + i(v \cdot \xi))^{-1} dz \in \mathcal{B}(V^{-1} \rightarrow V^1)$$

Kato's isomorphism:

$$\frac{\mathcal{P}(0)\mathcal{P}(\xi) + \mathcal{P}(0)^\perp \mathcal{P}(\xi)^\perp}{\sqrt{\text{Id} - (\mathcal{P}(\xi) - \mathcal{P}(0))^2}} =: \Phi(\xi) : \text{hydro. space}(\xi) \xrightarrow{\text{iso}} \ker(\mathcal{L})$$

$$\Phi(\xi) = \text{Id} + |\xi| \Phi^{(1)} + |\xi|^2 \Phi^{(2)}, \quad \Phi^{(j)} \in \mathcal{B}(V^{-1} \rightarrow V^1)$$

Rectified operator:

$$L(\xi) := \Phi(\xi)(\mathcal{L} - i(v \cdot \xi))\Phi(\xi)^{-1} \in \mathcal{B}(\ker \mathcal{L}) \sim \mathbb{C}^{(d+2) \times (d+2)}$$

Extra

Diagonalization of $L(\xi) = \Phi(\xi)^{-1} (\mathcal{L} - i(v \cdot \xi)) \Phi(\xi)$

Problem: $|\xi|^{-1} L(\xi) \xrightarrow{\xi \rightarrow 0}$ matrix with **non-simple** eigenvalues

Solution: Isotropy of $\mathcal{L} \Rightarrow$ block representation of $L(\xi)$:

$$L(\xi) = \begin{pmatrix} \lambda_{\text{inc}}(\xi) \text{Id} & 0 \\ 0 & |\xi| M(\xi) \end{pmatrix}, \quad \lambda_{\text{inc}}(\xi) = -\kappa_{\text{inc}} |\xi|^2 + \mathcal{O}(|\xi|^3),$$

in the decomposition

$$\{u \cdot v\mu \mid u \perp \xi\} \oplus \{\varrho\mu + \alpha\xi \cdot v\mu + e|v|^2\mu : (\varrho, \alpha, e) \in \mathbb{R}^3\}$$

where

$$M(\xi) = \begin{pmatrix} +ic & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -ic \end{pmatrix} + |\xi| \begin{pmatrix} -\kappa_{\text{osc}} & * & * \\ * & -\kappa_{\text{Fou}} & * \\ * & * & -\kappa_{\text{osc}} \end{pmatrix} + \mathcal{O}(|\xi|^2)$$

Conclusion: $L(\xi)$ diagonalizable + expansion of eigenprojector and eigenvalues:

$$\lambda_{\pm\text{osc}}(\xi) = \pm ic|\xi| - \kappa_{\text{osc}}|\xi|^2 + \mathcal{O}(|\xi|^3), \quad \lambda_{\text{Fou}}(\xi) = -\kappa_{\text{Fou}}|\xi|^2 + \mathcal{O}(|\xi|^3)$$

$$L(\xi) = \sum_{\star} \lambda_{\star}(\xi) P_{\star}(\xi), \quad P_{\star}(\xi) = P_{\star}^{(0)} + |\xi| P_{\star}^{(1)} + |\xi|^2 P_{\star}^{(2)}$$

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Consequence: expansion of $\Phi(\xi) \Rightarrow$ expansion of $(\mathcal{L} - i(v \cdot \xi))|_{\text{hydro. space}(\xi)}$:

$$(\mathcal{L} - i(v \cdot \xi))|_{\text{hydro. space}(\xi)} = \Phi(\xi)^{-1} L(\xi) \Phi(\xi)$$

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$$\mathcal{P}_{\star}(\xi) = \Phi(\xi)^{-1} P_{\star}(\xi) \Phi(\xi) = \mathcal{P}_{\star}^{(0)} + |\xi| \mathcal{P}_{\star}^{(1)} + |\xi|^2 \mathcal{P}_{\star}^{(2)}, \quad \mathcal{P}_{\star}^{(j)} : V^{-1} \rightarrow V^1$$